Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 2, 1998

MULTIVALUED SOLUTIONS OF THE SPATIAL PROBLEMS OF NONLINEAR DEFORMATION OF THIN CURVILINEAR RODS

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UDC 539.3

We have developed a finite-element model to study the spatial deformation of elastic rods at large displacements. A numerical algorithm for constructing multivalued nonlinear solutions in the presence of many bifurcation and limit points is formulated. Results of a study of the stability supercritical equilibrium forms of elastic rods, which were supported experimentally, are reported.

Beginning with the studies of Kirchhoff [1] and Clebsch [2], many publications have been devoted to the development of the theory of spatial bending of thin rods, for example, various versions of the nonlinear relations were discussed in [3–7] as applied to flexure in the region of large displacements and small elastic deformations. However, in practice the solution of the problems of nonlinear deformation of curvilinear rods causes some difficulties associated with the complexity of the description of finite rotations in a three-dimensional space. Exact analytical solutions can be derived only for a narrow class of problems [8] and, therefore, numerical methods of analysis acquire importance. The approaches to the construction of the solutions of a system of nonlinear differential equations that describe the finite displacements of curvilinear rods are considered [9, 10]. Ivanov and Ivanova [11, 12] developed the self-balanced discrepancy method as applied to calculation of stable equilibrium forms for cantilevered rods.

A number of papers [13–17] dealt with the development of finite-element models. It is worth noting that the application of the proposed numerical approaches to the solution of nonlinear static problems for rods was mainly illustrated by the construction of one-valued branches of the solutions without analysis of the stability of the equilibrium states found. At the same time, the solution of the nonlinear deformation problems for thin elastic bodies is known to be closely connected with stability problems and analysis of possible singular points on the curves of equilibrium states (deformation curves). The authors are not aware of studies devoted to nonlinear solutions with many bifurcation points and an analysis of the stability of equilibrium forms. It is clear that obtaining such solutions in the general case is possible with the use of numerical algorithms that are attributed to both the mechanics of discrete deformable systems and the methods of analysis of the solutions of the corresponding systems of nonlinear equations. The methods of constructing such solutions have not been examined adequately.

In the present paper, a finite element of a spatial deformable rod is proposed on the basis of the theory of kinematic groups [18]. Exact formulas for calculation of the coefficients of the first and second variations of the potential energy of the element are derived, which are used to formulate the conditions of equilibrium and stability. Problems of the construction of the solutions in the vicinity of singular points are considered. Two classical problems of flexure of a one-section beam and a circular ring under conservative loads are solved. It is shown that even in these known problems, there are many singular points and bifurcating solutions, which were not studied previously and which can be reproduced in experiments.

Chaplygin Siberian Aviation Institute, Novosibirsk 630051. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 2, pp. 141–149, March–April, 1998. Original article submitted July 11, 1996.

 0021-8944/98/3902-0282 $20.00

1. To formulate the finite-element model of a spatial rod, we use the propositions of the theory of kinematic groups [18], which is intended to construct discrete analogs of nonlinearly deformable bodies whose displacements and rotations are not restricted. We shall use the notation adopted in [18]. In addition, we shall use summation over repeat indices, unless otherwise specified.

We select two nodal points (0 and 1) on the axial line of a rod (the lines of centoids of the transverse cross sections) and place, at each of them, two orthogonal vectors [$\mathbf{d}_{0j}$ and $\mathbf{d}_{1j}$ ($j = 1$ and 2)] which lie in the plane of transverse cross sections. The thus-formed kinematic group is characterized by the following three metric tensors, the first of which is a scalar:

$$a_{11} = \mathbf{r}_1\mathbf{r}_1, \quad b_{1mp} = \mathbf{r}_1\mathbf{d}_{mp}, \quad c_{mpnq} = \mathbf{d}_{mp}\mathbf{d}_{nq} \quad (m,\ n = 0,\ 1 \text{ and } p,\ q = 1,\ 2).$$

Here $\mathbf{r}_1 = \mathbf{R}_1 - \mathbf{R}_0$ (R_m are the nodal values of the radius-vector of the rod's axial line).

We shall present relations for the kinematic group, which are necessary for subsequent manipulations. The strain tensors of the group are of the form

$$\varepsilon_{11} = \frac{1}{2}(\mathbf{r}_1^\vee\mathbf{r}_1^\vee - \mathbf{r}_1\mathbf{r}_1), \quad \vartheta_{1mp} = \frac{1}{2}(\mathbf{r}_1^\vee\mathbf{d}_{mp}^\vee - \mathbf{r}_1\mathbf{d}_{mp}), \quad \nu_{mpnq} = \frac{1}{2}(\mathbf{d}_{mp}^\vee\mathbf{d}_{nq}^\vee - \mathbf{d}_{mp}\mathbf{d}_{nq})$$
$$(m,\ n = 0,\ 1 \text{ and } p,\ q = 1,\ 2),$$

where the superscript v denotes the quantities attributed to the deformed state.

For the first strain tensor, two variations can be determined, whereas there are arbitrary-order variations for the second and third tensors:

$$\delta\varepsilon_{11} = \mathbf{r}_1^\vee(\delta\mathbf{R}_1^\vee - \delta\mathbf{R}_0^\vee), \quad \delta^2\varepsilon_{11} = (\delta\mathbf{R}_1^\vee - \delta\mathbf{R}_0^\vee)^2, \quad \delta^k\vartheta_{1mp} = \frac{1}{2}(\mathbf{r}_1^\vee\delta^k\mathbf{d}_{mp}^\vee + k\delta\mathbf{r}_1^\vee\delta^{k-1}\mathbf{d}_{mp}^\vee),$$
$$\delta^k\nu_{mpnq} = \frac{1}{2}\sum_{s=0}^{k} C_k^s\delta^s\mathbf{d}_{mp}^\vee\delta^{k-s}\mathbf{d}_{nq}^\vee \qquad (k = 1,\ 2, \ldots). \tag{1.1}$$

Here C_k^s are binominal coefficients. For subsequent manipulations, it suffices to confine ourselves to the case $k = 1$ and 2, i.e., to consider the calculation of only two first variations of the strain tensors of the group.

We assume that the added vectors at the nodes remain unit and orthogonal in the process of group deformation, i.e., the condition $\nu_{mpmq} = 0$ is satisfied (no summation over m). In this case, the variations of the added vectors in (1.1) are given by the following formulas (no summation over m):

$$\delta\mathbf{d}_{mp}^\vee = \delta\boldsymbol{\omega}_m \times \mathbf{d}_{mp}^\vee, \quad \delta^2\mathbf{d}_{mp}^\vee = \delta\boldsymbol{\omega}_m \times (\delta\boldsymbol{\omega}_m \times \mathbf{d}_{mp}^\vee) \quad (m = 0,\ 1 \text{ and } p = 1,\ 2). \tag{1.2}$$

Here ω_m are the nodal values of the rotation vector. We shall use the expansion of the vector in the basis $\mathbf{e}_i$ of the Cartesian coordinate system x_i ($i = 1$, 2, and 3):

$$\mathbf{R}_m = \mathbf{e}_i x_{mi}, \quad \mathbf{d}_{mp} = \mathbf{e}_i\lambda_{mpi}, \quad \omega_m = \mathbf{e}_i\omega_{mi} \quad (m = 0,\ 1,\ \ p = 1,\ 2, \text{ and } i = 1,\ 2,\ 3). \tag{1.3}$$

It follows from relations (1.1)–(1.3) that three variations of the Cartesian coordinates and three variations of the components of the rotation vector correspond to each node. Thus, the possible states of the kinematic groups considered are characterized by a 12-component vector of the variations of the generalized coordinates:

$$\delta\mathbf{q}^t = [\delta\mathbf{q}_0^t, \delta\mathbf{q}_1^t], \quad \delta\mathbf{q}_m^t = [\delta x_{m1}^\vee, \delta x_{m2}^\vee, \delta x_{m3}^\vee, \delta\omega_{m1}, \delta\omega_{m2}, \delta\omega_{m3}].$$

The strain state of the groups is determined by six independent components which form the vector of generalized elastic displacements: $\mathbf{u}^t = [\varepsilon_{11}, \vartheta_{101}, \vartheta_{102}, \vartheta_{111}, \vartheta_{112}, \theta]$ ($\theta = \nu_{1102} - \nu_{1201}$ is a parameter that characterizes the mutual rotations of the planes of transverse cross sections at the nodes of the group).

2. We shall consider the strain relations for the finite element of a rod that are associated with this kinematic group. If the nodes on the axial line of the rod are chosen to be sufficiently close, the element will be shallow relative to the secant passing through its nodes. Here the strains of the element can be expressed via the strains of the group:

$$\varepsilon = \varepsilon_{11}/l^2, \quad æ_p = N_m''\vartheta_{1mp}, \quad \chi = \theta/l \quad (p = 1,\ 2;\ \ m = 0,\ 1), \tag{2.1}$$

$$N_0 = 2s(s-l)^2/l^3, \quad N_1 = 2s^2(s-l)/l^3, \quad 0 \leqslant s \leqslant l.$$

Here l is the length of the element and the derivative with respect to the coordinate s is primed.

We write the potential deformation energy of the finite element in the form

$$\Pi = \frac{1}{2}\int_0^l (EF\varepsilon^2 + EI_{pq}æ_p æ_q + GJ\chi^2)\,ds, \tag{2.2}$$

where EF, EI_{pq}, and GJ are the tensile, flexural, and torsional rigidities of the rod. Substituting (2.1) into (2.2) and integrating, we obtain the following expression for the potential energy as a quadratic form relative to the components of the vector of generalized elastic displacements:

$$\Pi = (1/2)\mathbf{u}^t\mathbf{K}\mathbf{u}.$$

Here $\mathbf{K}$ is the symmetric stiffness matrix whose nonzero components are of the form

$$K_{11} = EF/l^3, \quad K_{22} = 16EI_{11}/l^3, \quad K_{23} = 16EI_{12}/l^3, \quad K_{24} = (1/2)K_{22},$$
$$K_{25} = (1/2)K_{23}, \quad K_{33} = 16EI_{22}/l^3, \quad K_{34} = (1/2)K_{23}, \quad K_{35} = (1/2)K_{33},$$
$$K_{44} = K_{22}, \quad K_{45} = K_{23}, \quad K_{55} = K_{33}, \quad K_{66} = GJ/l.$$

The first and second variations of the potential energy are, respectively, the linear and quadratic forms of the form $\delta\Pi = \delta\mathbf{q}^t\mathbf{g}$ and $\delta^2\Pi = \delta\mathbf{q}^t\mathbf{H}\delta\mathbf{q}$, where $\mathbf{g}$ and $\mathbf{H}$ are the gradient and the Hess matrix of the element, respectively, which are calculated by the formulas

$$\mathbf{g} = \mathbf{u}'\mathbf{P}, \quad \mathbf{P} = \mathbf{K}\mathbf{u}, \quad \mathbf{H} = \mathbf{u}'\mathbf{K}(\mathbf{u}')^t + P_i\mathbf{u}_i'' \qquad (i = 1, \ldots, 6).$$

Here $\mathbf{u}'$ and $\mathbf{u}_i''$ are matrices that establish the dependence of the first and second variations of the components of the vector of generalized elastic displacements on the variations of the generalized coordinates of the kinematic group. The derivatives below, which determine the matrices $\mathbf{u}'$ and $\mathbf{u}_i''$, are found as the coefficients at the variations of generalized coordinates after the variations of the components of the vector $\mathbf{u}$ are derived using relations (1.1)–(1.3).

The nonzero components of the matrix $\mathbf{u}'$ are calculated according to the formulas (no summation over m)

$$\frac{\partial\varepsilon_{11}}{\partial x_{mi}} = b_m b_n x_{ni}, \quad \frac{\partial\vartheta_{1mp}}{\partial x_{ni}} = \frac{1}{2} b_n\lambda_{mpi}, \quad \frac{\partial\vartheta_{1mp}}{\partial\omega_{mi}} = \frac{1}{2} e_{ijk} b_n x_{nk}\lambda_{mpj},$$
$$\frac{\partial\theta}{\partial\omega_{0i}} = \frac{1}{2} e_{ijk} D_{kj}, \quad \frac{\partial\theta}{\partial\omega_{1i}} = \frac{1}{2} e_{ijk} D_{jk},$$
$$D_{jk} = \lambda_{11j}\lambda_{02k} - \lambda_{12j}\lambda_{01k}, \quad b_0 = -b_1 = -1 \quad (i,\, j,\, k = 1,\, 2,\, 3, \quad m, n = 0,\, 1, \text{ and } \quad p = 1,\, 2),$$

where $e_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j)\mathbf{e}_k$ are the Levi-Civita symbols. Here and below, the superscript v is omitted for brevity.

The nonzero components of the matrices $\mathbf{u}_i''$ are determined by the formulas (no summation over i and m)

$$\frac{\partial^2\varepsilon_{11}}{\partial x_{mi}\partial x_{ni}} = b_m b_n, \quad \frac{\partial^2\vartheta_{1mp}}{\partial x_{nk}\partial\omega_{mi}} = \frac{1}{2} e_{ijk} b_n\lambda_{mpj},$$
$$\frac{\partial^2\vartheta_{1mp}}{\partial\omega_{mi}\partial\omega_{mj}} = \frac{1}{4} b_n x_{nk}\lambda_{mpl}(\delta_{kj}\delta_{il} + \delta_{ki}\delta_{jl} - 2\delta_{kl}\delta_{ij}), \quad \frac{\partial^2\theta}{\partial\omega_{mi}\partial\omega_{mj}} = \frac{1}{4}(D_{ij} + D_{ji} - 2\delta_{ij}D_{kk}),$$
$$\frac{\partial^2\theta}{\partial\omega_{0i}\partial\omega_{1j}} = \frac{1}{2}(\delta_{ij}D_{kk} - D_{ij}) \qquad (i,\, j,\, k,\, l = 1,\, 2,\, 3; \;\; m, n = 0,\, 1; \;\; p = 1,\, 2)$$

(δ_{ij} is the Kronecker symbol).

The relations obtained enable one to calculate the gradient and Hess matrix of the finite element of the rod.

3. We shall consider the iterative method of finding the spectrum of solutions of nonlinear equilibrium problems for a discrete system under conservative loads. We assume that the variations of the potential deformation energy Π of a discrete system (an ensemble of finite elements) depend on N variations of the generalized coordinates: $\delta q_1, \dots, \delta q_N$, and the variation of the potential of external forces W depends on $N+1$ variations: $\delta q_1, \dots, \delta q_N, \delta q_{N+1}$. By δq_{N+1}, we mean the variation of the load parameter.

The variational equilibrium equation is of the following form (equation of the Newton–Rafson method with a varying load parameter) at the kth iteration of some extension step:

$$h_{ij}^{k-1} \delta q_j^k + g_i^{k-1} = 0. \tag{3.1}$$

Here $h_{ij}^k = (h_{ij}^\Pi + h_{ij}^W)^k$, $g_i^k = (g_i^\Pi + g_i^W)^k$ ($i = 1, \dots, N$; $j = 1, \dots, N+1$); g_i^k and h_{ij}^k are the components of the gradient and the Hess matrix of the total potential energy of an ensemble of finite elements; the superscripts Π and W refer to the potential deformation energy and the potential of external forces, respectively; the values of g_i^W and h_{ij}^W are determined by the formulas $g_i^W = \partial W / \partial q_i$ and $h_{ij}^W = \partial^2 W / \partial q_i \partial q_j$ for the general case.

System (3.1) consists of N equations for $N+1$ unknowns. We construct the general solution of this system assuming that the N-order square matrix $||h_{ij}^k||$ is not degenerate. We seek the solution of Eqs. (3.1) which belongs to the Euclidean space R_{N+1} with the norm $\delta s^k = (\delta q_i^k \delta q_i^k)^{1/2}$ ($i = 1, \dots, N+1$) in the following form (no summation over k):

$$\delta q_j^k = \delta q_{N+1}^k x_j^k + y_j^k \qquad (j = 1, \dots, N), \tag{3.2}$$

where x_j^k and y_j^k are obtained from the systems (i, $j = 1, \dots, N$)

$$h_{ij}^{k-1} x_j^k + h_{i,N+1}^{k-1} = 0; \tag{3.3}$$

$$h_{ij}^{k-1} y_j^k + g_i^{k-1} = 0. \tag{3.4}$$

We have $g_i^0 = 0$ at the stationarity points of the total potential energy (equilibrium states) and $y_i^1 = 0$ according to (3.4). A transition from one to another stationary point is performed over the norm $\delta s^1 = (\delta q_i^1 \delta q_i^1)^{1/2}$ ($k = 1$ and $i = 1, \dots, N+1$) with subsequent refinement of the solution in the plane orthogonal to the vector $\delta \mathbf{q}^1$, i.e., $\delta q_i^1 \delta q_i^k = 0$ ($k > 1$ and $i = 1, \dots, N+1$) [19, 20]. With allowance for (3.2), we obtain the formulas required to find δq_{N+1}^k:

$$\delta q_{N+1}^1 = \pm \delta s^1 / (1 + x_i^1 x_i^1)^{1/2} \quad \text{for} \quad k = 1; \tag{3.5}$$

$$\delta q_{N+1}^k = -x_i^1 y_i^1 / (1 + x_i^1 x_i^k) \quad \text{for} \quad k > 1. \tag{3.6}$$

The sign in (3.5) determines the direction of the continuation of the solution. In moving along the curve of equilibrium states in a prescribed direction, the sign in (3.5) is chosen at each step of continuation from the condition that the angle between the vectors $\delta \mathbf{q}^1 = [\delta q_1^1, \dots, \delta q_{N+1}^1]^t$ calculated at two neighboring steps does not exceed $\pi/2$.

The new positions of the metric vectors of the kinematic group can be found on the basis of the calculated variations of the generalized coordinates using formulas of [18] (no summation over k and m):

$$x_{mi}^k = x_{mi}^{k-1} + \delta x_{mi}^k,$$

$$\lambda_{mps}^k = \lambda_{mps}^{k-1} + A_m^k e_{ijs} \lambda_{mpj}^{k-1} \delta \omega_{mi}^k + B_m^k (\lambda_{mpi}^{k-1} \delta \omega_{mi}^k \delta \omega_{ms}^k - \lambda_{mps}^{k-1} \delta \omega_{mi}^k \delta \omega_{mi}^k)$$

$$(i,\ j,\ s = 1,\ 2,\ 3, \quad m = 0,\ 1, \ \text{and}\ p = 1,\ 2),$$

$$A_m^k = \sin \delta \omega_m^k / (\delta \omega_m^k), \quad B_m^k = (1 - \cos \delta \omega_m^k) / (\delta \omega_m^k)^2.$$

In the case of singular points on the deformation curve, the square matrix $||h_{ij}^0||$ is degenerate and it is necessary to distinguish singular points of two types, depending on whether system (3.3) is compatible or not. If the system is not compatible, the general solution (3.1) for $k = 1$ has the form

$$\delta q_j^1 = \mu_i f_{ji}, \quad \delta q_{N+1} = 0 \qquad (i = 1, \dots, l < N \ \text{and}\ j = 1, \dots, N),$$

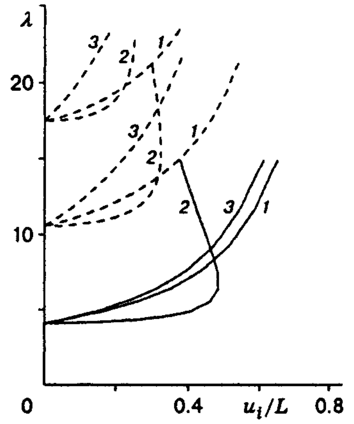

Fig. 1

TABLE 1

n	Critical-point number 1	2	3
	λ_*		
2	4.904	—	—
4	4.198	12.032	26.589
6	4.096	11.002	19.233
8	4.062	10.671	18.065
10	4.047	10.523	17.529
12	4.038	10.444	17.224
Solution [22]	4.01	10.24	—

where μ_i are arbitrary factors and f_{ji} are the linearly independent nontrivial solutions of the homogeneous system of equations $h^0_{ij} f_{jr} = 0$ ($i, j = 1, \ldots, N$ and $r = 1, \ldots, l$). For convenience, we assume f_{jr} to be chosen such that the condition $f_{jr} f_{jp} = \delta_{pr}$ holds. Thus, the column vectors of the matrix $||f_{jr}||$ determine the possible directions of continuing the solution in the vicinity of a singular point. Here a transition to the bifurcating solutions is performed with respect to the norm

$$\delta s^1 = (\mu_r \mu_r)^{1/2} \qquad (k = 1 \text{ and } r = 1, \ldots, l) \tag{3.7}$$

with further refinement according to formulas (3.2) and (3.6).

If $\det||h^0_{ij}|| = 0$ and system (3.3) is compatible, which is determined by the condition $f_{jr} h^0_{j,N+1} = 0$ [20], the general solution is written in the form

$$\delta q^1_j = \delta q^1_{N+1} x^1_j + \mu_r f_{jr} \qquad (j = 1, \ldots, N, \text{ and } r = 1, \ldots, l < N).$$

In this case, the solutions at any point in the neighborhood of a singular point can be obtained by both formulas (3.5) and (3.6) and by formulas (3.7) and (3.6). We note that at $l = 1$ the points of the types considered are known as the limit and the bifurcation point [21].

The stability problem for the equilibrium states obtained is solved based on the Sylvester criterion on the positive definiteness of the matrix $||h^0_{ij}||$. Such information can be easily obtained by directly using the Gauss method in solving systems (3.3) and (3.4).

4. We shall consider the application of the developed algorithm in analysis of the stability and post-critical equilibrium forms of a cantilevered one-section beam loaded by a concentrated force P at its free end in the plane of largest rigidity. The following characteristics were adopted for the beam: $L/b = 10$, $b/h = 40$, and $\nu = 0.3$ (L is the beam length, b and h are the width and thickness of the cross section, and ν is the Poisson ratio).

Table 1 gives the dependences of the critical load parameter $\lambda_* = P_* L^2/\sqrt{GJ \cdot EI}$ (EI is the smallest flexural rigidity of the beam) on the number of finite elements n for the first three critical points. For comparison, we present the analytical solution of [22] derived under the assumption that the beam is not deformable in the subcritical state.

The post-critical behavior was studied with the use of a uniform grid consisting of 10 finite elements. Figure 1 shows the nonlinear deformation characteristics where curves 1–3 correspond to the displacements u_i ($i = 1$, 2, and 3) of the beam's free end in the direction of the x_i axes, and the solid and dashed curves indicate the stable and unstable states, respectively. It follows from calculations that, having lost stability, the displacement u_2, which corresponds to the out-of-plane deformation of the beam, increases to a certain limit and then decreases. For example, in bifurcating the solution from the first critical point, the maximum displacement is $u_2^{\max} = 0.485L$.

Figures 2 and 3 show, on a real scale, the forms of beam deformation in the post-critical region, which correspond to the solution bifurcating from the first two bifurcation points.

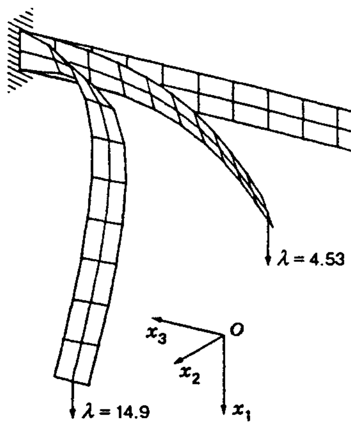

Fig. 2

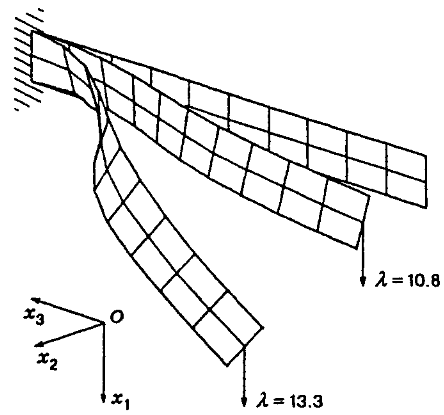

Fig. 3

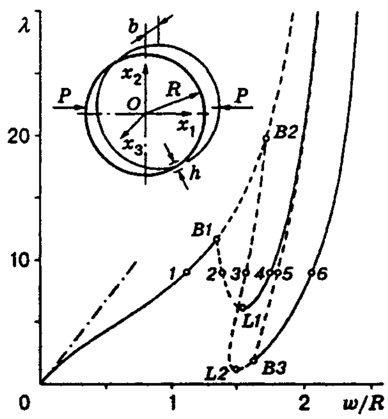

Fig. 4

We shall consider the spectrum of nonlinear solutions in the problem of a circular ring of a narrow rectangular cross section compressed by two radial forces P. The ring is characterized by the following parameters: $R/b = 10$, $b/h = 40$, and $\nu = 0.3$ (R is the ring radius). The deformation forms that are symmetric relative to the plane Ox_1x_3 were examined. Therefore, discretization was performed for half the ring using a uniform grid comprising 28 finite elements. As additional numerical studies have shown, the adopted degree of discretization is sufficient to obtain nonlinear solutions with high accuracy in a wide range of form variations of the ring. No restrictions are imposed on the possibility of self-crossing the ring.

Figure 4 shows the nonlinear deformation characteristic, where w is the deflection at the point of force application, EI is the smallest flexural rigidity, $\lambda = PR^2/EI$ is a load parameter, the dot-and-dashed curve refers to the linear solution of the problem, and the solid and dashed curves refer to the stable and unstable equilibrium states, respectively. We note that the nonlinear solution at $0 \leqslant w/R \leqslant 1$ was obtained, for example, by Popov [23] for an unextensional ring. The study of the problem in a spatial formulation with the use of the developed algorithm allowed us to reveal the previously unknown singular points and to study the bifurcating solutions. The basic branch of the equilibrium states, which corresponds to the monotonically increasing load, is characterized by plane symmetrical flexural forms of equilibrium of the ring (Fig. 5a). As the load parameter grows to $\lambda_* = 11.75$, the possibility of branching the equilibrium forms (bifurcation point B1) appears. The existence of the point B1 for an unextensional ring was shown by Seide and Albano [24] in an analytical manner, and it was found that $\lambda_* = 11.9$, which differs little from the result obtained using the developed algorithm. The nonlinear solution which bifurcates from the basic branch at the point B1 and contains the limiting point L1 was analyzed by Kuznetsov and Soinikov [25]. The equilibrium forms for a ring

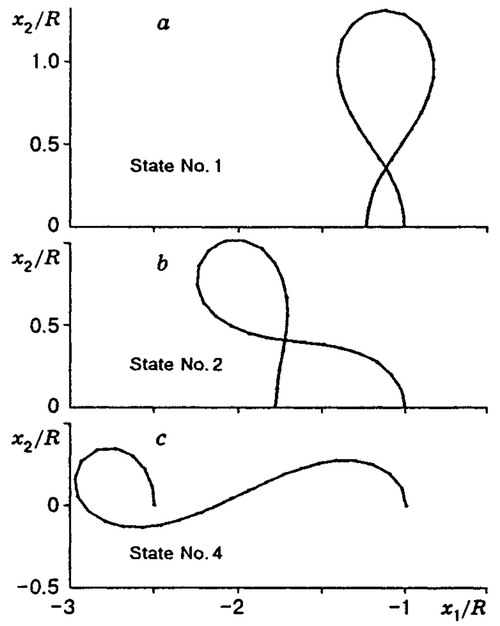

Fig. 5

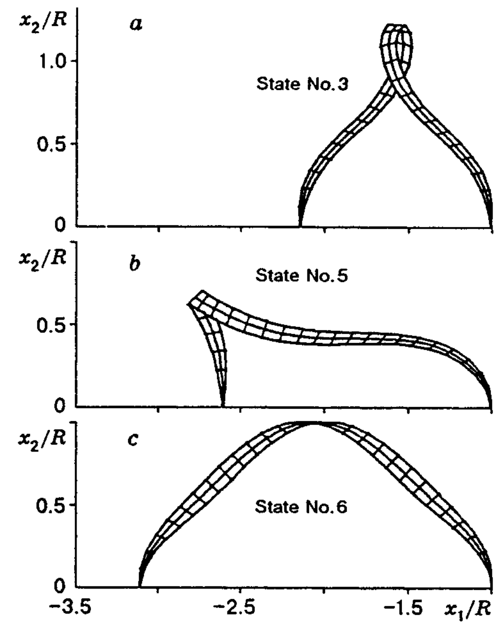

Fig. 6

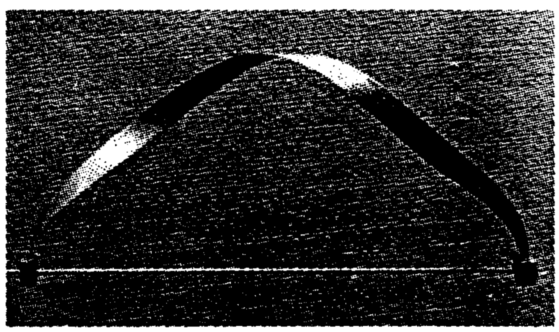

Fig. 7

which correspond to the branch B1–L1 are plane and flexurable, but they do not possess the symmetry of deformation (Fig. 5b and c).

Calculations have shown that there is one more bifurcation point B2 on the basic branch with $\lambda_* = 19.8$ at which a transition to unstable spatial configurations of the ring is possible. This transition is connected with the rotation of the ring about the Ox_2 axis (Fig. 6a). An analysis of the nonlinear solution that bifurcates at the point B2 made it possible to find the limit point L2 ($\lambda_* = 1.27$) and the bifurcation point B3 ($\lambda_* = 1.97$) such that having passed through it the spatial forms of equilibrium become stable (Fig. 6c). The forms of deformation of the ring that correspond to the branch B2–L2–B3 are flexural-torsional, the projection of the axial line of the ring onto the plane Ox_1x_2 being a symmetric curve.

Bifurcation of the solution at the point B3 means a transition to spatial unstable flexural–torsional forms which do not possess any symmetry of deformation (Fig. 6b).

The forms of equilibrium for a ring were reproduced in the experiment on a thin celluloid model. Figure 7 shows a photograph of the deformed state of the ring that corresponds to Fig. 6b (state 6).

In concluding, we note that the finite-element model of a rod that we constructed and the numerical algorithm developed allow one to solve effectively the complicated problems of spatial deformation of curvilinear rods in the presence of many singular points, to examine the multiple bifurcation of solutions, and to analyze the stability of the equilibrium states found.

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